1 Algebraic Notation

1. Algebra is so much like arithmetic that all that you know about addition, subtraction, multiplication, and division, the signs that you have been using and the ways of working out problems, will be very useful to you in this study. There are two things the introduction of which really makes all the difference between arithmetic and algebra. One of these is the use of *letters to represent numbers*, and you will see in the following exercises that this change makes the solution of problems much easier.

1.1 Problems

Exercise I.

Illustrative Example. The sum of two numbers is 60, and the greater is four times the less. What are the numbers?

Solution.

Let	$x=$	the less number;
then	$4x=$	the greater number,
and	$4x+x=60$,	
or	$5x=60$;	
therefore	$x=12$,	
and	$4x=48$.	The numbers are 12 and 48.

1. The greater of two numbers is twice the less, and the sum of the numbers is 129. What are the numbers?
   Solution: $x + \underbrace{2 \cdot x}_{\text{greater}} = 129 \overset{:3}{\Leftrightarrow} x = 43.\ 2x = 86.$
   Check: $43 + 86 = 129$✓ 43;86
2. A man bought a horse and carriage for \$500, paying three times as much for the carriage as for the horse. How much did each cost?
   Solution: $\underbrace{x}_{\text{horse}} + \underbrace{3x}_{\text{carriage}} = 500 \Leftrightarrow 4x = 500 \overset{:4}{\Leftrightarrow} x = 125.\ 3x = 375.$
   Check: $125 + 375 = 500$✓ Horse: \$125; carriage: \$375.

3. Two brothers, counting their money, found that together they had \$186, and that John had five times as much as Charles. How much had each?

   Solution: $\underbrace{x}_{\text{Charles}} + \underbrace{5x}_{\text{John}} = 186 \Leftrightarrow 6x = 186 \overset{:6}{\Leftrightarrow} x = 31.$

   Check: $31 + 5 \cdot 31 = 186$✓ $\underline{\underline{\text{Charles: \$31; John: \$155}}}$

4. Divide the number 64 into two parts so that one part shall be seven times the other.

   Solution: $64 = x + 7x \Leftrightarrow 8x = 64 \overset{:8}{\Leftrightarrow} x = 8.\ 7x = 56.$

   Check: $56 + 8 = 64$✓ $\underline{\underline{8; 56}}$

5. A man walked 24 miles in a day. If he walked twice as far in the forenoon as in the afternoon, how far did he walk in the afternoon?

   Solution: $24 = \underbrace{2x}_{\text{forenoon}} + \underbrace{x}_{\text{afternoon}} \Leftrightarrow 24 = 3x \overset{:3}{\Leftrightarrow} x = 8.$ Check: $24 = 2 \cdot 8 + 8$✓

   $\underline{\underline{\text{In the afternoon, he walked 8 miles.}}}$

6. For 72 cents Martha bought some needles and thread, paying eight times as much for the thread as for the needles. How much did she pay for each?

   Solution: $72 = \underbrace{8x}_{\text{thread}} + \underbrace{x}_{\text{needles}}.\ 72 = 9x \overset{:9}{\Leftrightarrow} x = 8.$ Check: $72 = 8 \cdot 8 + 8 = 64 + 8$✓

   $\underline{\underline{\text{Thread: 64 cents; needles: 8 cents.}}}$

7. In a school there are 672 pupils. If there are twice as many boys as girls, how many boys are there?

   Solution: $672 = \underbrace{2x}_{\text{boys}} + \underbrace{x}_{\text{girls}} \Leftrightarrow 672 = 3x \overset{:3}{\Leftrightarrow} x = 224.\ 2x = 2 \cdot 224 = 448.$

   Check: $448 + 224 = 672$✓ $\underline{\underline{\text{There are 448 boys and 224 girls.}}}$

Illustrative Example. If the difference between two numbers is 48, and one number is five times the other, what are the numbers?

Solution.

Let	$x =$ the less number;
then	$5x =$ the greater number,
and	$5x - x = 48,$
or	$4x = 48;$
therefore	$x = 12,$
and	$5x = 60.$

The numbers are 12 and 60.

8. Find two numbers such that their difference is 250 and one is eleven times the other.

   Solution: $250 = \underbrace{11x - x}_{\text{difference}} \Leftrightarrow 250 = 10x \overset{:10}{\Leftrightarrow} x = 25.\ 11 \cdot 25 = 275.$

   Check: $275 - 25 = 250$✓ $\underline{\underline{275; 50}}$

9. James gathered 12 quarts[1] of nuts more than Henry gathered. How many did each gather if James gathered three times as many as Henry?

   Solution: The base reference is Henry, he gathered x quarts. From James we know that he gathered "12 quarts of nuts more than Henry" what equals $x + 12$, and we know, that Henry gathered in total "three times as many as Henry" what equals $3x$. From these information we get our mathematical equation: The number of nuts James has in total is described two times, both descriptions describe the identical amount of nuts, so that we can set them equal.

$$\left.\begin{aligned} &\text{James: } x + 12 \quad \text{or} \quad 3x \\ &\text{Henry: } x \end{aligned}\right\} x + 12 = 3x$$

$$\begin{aligned} \text{12 quarts of nuts more than Henry} &= \text{three times as many as Henry} \\ x + 12 &= 3x \quad \mid -x \\ 12 &= 2x \quad \mid :2 \\ x &= 6 \end{aligned}$$

   Insert x in the equation and verify if the statement is true:

$$\begin{aligned} x + 12 &= 3x \\ 6 + 12 &= 3 \cdot 6 \\ 18 &= 18 \checkmark \end{aligned}$$

   $\underline{\underline{\text{Henry: 6 quarts; James: 18 quarts.}}}$

10. A house costs \$2880 more than a lot[2] of land, and five times the cost of the lot equals the cost of the house. What was the cost of each?

    Solution: With x representing the lot, the cost for the house is $2880 + x$. This is equal to "five times the cost of the lot", i.e. $5x$.

[1] "The quart (abbreviation qt.) is an English unit of volume equal to a quarter gallon. It is divided into two pints or four cups.[...]" (https://en.wikipedia.org/wiki/Quart)

[2] "In real estate, a lot or plot is a tract or parcel of land owned or meant to be owned by some owner(s).[...]"(https://en.wikipedia.org/wiki/Land_lot)

$2880 + x = 5x \overset{-x}{\Leftrightarrow} 2880 = 4x \overset{:4}{\Leftrightarrow} x = 720$. The cost for the lot is \$720. The price for the house is $5x$ resp. $5 \cdot 720 = 300$. Check: $(2880 + 720) = 5 \cdot 720 \Leftrightarrow 3600 = 3600$✓
Lot: \$720; house: \$3600.

11. Mr. A. is 48 years older than his son, but he is only three times as old. How old is each?

    Solution: Let x be the age of the son. We get told two times how old the father is, thus we can set equal these information as both describe the identical number of years: "48 years older than his son" $(48+x)$ and "he is only three times as old" $(3x)$. Hence $48 + x = 3x$. $48 = 2x \overset{:2}{\Leftrightarrow} x = 24$. Check: $48 + 24 = 3 \cdot 24 \Leftrightarrow 72 = 72$✓
    Father: 72 years, son: 24 years.

12. Two farms differ by 250 acres, and one is six times as large as the other. How many acres in each?

    Solution: Here, the word "differ" refers to a difference (subtraction). The result of the difference is 250. The bigger farm is six times as large $(\cdot 6)$ as the smaller one (x). $250 = 6x - x$. $250 = 5x \overset{:5}{\Leftrightarrow} 50 = x$. The smaller farm contains 50 acres. The bigger one contains $6 \cdot x = 300$ acres. Check: $250 = 6x - x \Leftrightarrow 250 = 300 - 50$✓
    Big farm: 300 acres, small farm: 50 acres.

13. William paid eight times as much for a dictionary as for a rhetoric.[3] If the difference in price was \$6.30, how much did he pay for each?

    Solution: We can convert the price in dollars to cents, \$6.30 are ¢630. With x being the price for the rhetoric the price for the dictionary is $8x$. Then we have the difference in price which results in 6.30 dollars resp. in 630 cents: $8x - x = 630 \Leftrightarrow 7x = 630 \overset{:7}{\Leftrightarrow} x = 90$. Thus the price for the rhetoric is ¢90 resp. \$0.90, the price for the dictionary is ¢720 resp. \$7.20 $(8 \cdot 90 = 720)$. Check: $8 \cdot 90 - 90 = 720 - 90 = 630$✓
    Dictionary: \$7.20, rhetoric: \$0.90.

14. The sum of two numbers is 4256, and one is 37 times as great as the other. What are the numbers?

    Solution: $4256 = 37x + x$. $4256 = 38x$. $4256 : 38 = 112$, the smaller number is 112. $37 \cdot x = 37 \cdot 112 = 4144$, the bigger number is 4144. Check: $4144 + 112 = 4256$✓
    4144; 112

15. Aleck has 48 cents more than Arthur, and seven times Arthur's money equals Aleck's. How much has each?

[3] "[...] **3** A textbook treating of discourse; especially, written discourse. [...]", an excerpt of the definition for the word "rhetoric" (the third of five meanings), taken from *The New International Webster's Comprehensive Dictionary of the English Language, Enyclopedic Edition, 2003 Edition [Bell Vista]*, page 1080.

Solution: Let x be the money of Arthur. The amount of Aleck's money is described twice: "Aleck has 48 cents more than Arthur" $(x + 48)$ and "seven times Arthur's money" $(7x)$. Describing the identical amount of money, we can set equal these information to get our initial equation: $x + 48 = 7x \overset{-x}{\Leftrightarrow} 48 = 6x \overset{:6}{\Leftrightarrow} x = 8$. Hence, Arthur has 8 cents and Aleck has $x + 48 = 56$ cents. Check: $48 + 8 = 7 \cdot 8$✓
Arthur: 8 cents, Aleck: 56 cents.

16. The sum of the ages of a mother and daughter is 32 years, and the age of the mother is seven times that of the daughter. What is the age of each?

    Solution: $32 = \overbrace{7x}^{\text{mother}} + \overbrace{x}^{\text{daughter}}$. $32 = 8x \overset{:8}{\Leftrightarrow} x = 4$. The daughter is 4 years old. The mother is 7 times as old as the daughter, $7 \cdot x = 7 \cdot 4 = 28$ years old. Check: $32 = 7 \cdot 4 + 4$✓
    Mother: 28 years; daughter: 4 years.

17. John's age is three times that of Mary, and he is 10 years older. What is the age of each?

    Solution: Let x be the age of Mary. Then John's age is $3x$ ("three times that of Mary") resp. $(x + 10)$ ("he is 10 years older"). The latter statements describe the identical number of years, so we can set them equal to get our initial equation: $3x = x + 10 \overset{-x}{\Leftrightarrow} 2x = 10 \overset{:2}{\Leftrightarrow} x = 5$. Check: $3 \cdot 5 = 5 + 10$✓
    Age of Mary: 5 years; Age of John: 15 years.

Exercise 2.

Illustrative Example. There are three numbers whose sum is 96; the second is three times the first, and the third is four times the first. What are the numbers?

Solution.

$$
\begin{aligned}
\text{Let} \qquad x &= \text{first number,} \\
3x &= \text{second number,} \\
4x &= \text{third number.} \\
x + 3x + 4x &= 96 \\
8x &= 90 \\
x &= 12 \\
3x &= 36 \\
4x &= 48
\end{aligned}
$$

The numbers are 12, 36, and 48.

1. A man bought a hat, a pair of boots, and a necktie for \$7.50; the hat cost four times as much as the necktie, and the boots cost five times as much as the necktie. What was the cost of each?

Solution: Here, the price for the necktie is our initial value x. The hat costs four times as much as the necktie ($4x$) and the boots cost five times as much ($5x$), all together cost \$7.50. $7.50 = x + 4x + 5x \Leftrightarrow 7.50 = 10x \overset{:10}{\Leftrightarrow} 0.75 = x$. The necktie costs \$0.75, the hat costs ($4 \cdot 0.75 = 3$) \$3, the boots cost ($5 \cdot 0.75 = 3.75$) \$3.75. Check: $0.75 + 3 + 3.75 = 7.5\checkmark$ Necktie: \$0.75; hat: \$3; boots: \$3.75.

2. A man travelled 90 miles in three days. If he travelled twice as far the first day as he did the third, and three times as far the second day as the third, how far did he go each day?

   Solution: The reference point resp. our initial value is given by the third day, on the *third day* he travelled x miles. On the *first day* he travelled "twice as far [...] as he did the third [day]" ($2x$), and on the *second day* "three times as far [...] as [on] the third" ($3x$). Hence $90 = x + 2x + 3x \Leftrightarrow 90 = 6x \overset{:6}{\Leftrightarrow} x = 15$. On the first day he travels 15 miles, on the second 30 miles ($2 \cdot 15$), and on the third he travels 45 miles ($4 \cdot 15$). Check: $15 + 30 + 45 = 90\checkmark$ 3rd day: 15 miles; 2nd day: 30 miles; 1st day: 45 miles.

3. James had 30 marbles. He gave a certain number to his sister, twice as many to his brother, and had three times as many left as he gave his sister. How many did each then have?

   Solution: The "certain number" is x, "twice as many" is $2x$. "and had tree times as many left as he gave his sister" refers to the initial value, so the latter is $3 \cdot x$. $30 = x + 2x + 3x \Leftrightarrow 30 = 6x \overset{:6}{\Leftrightarrow} x = 5$. $2x = 10$, $3x = 15$. Check: $30 = 5 + 10 + 15\checkmark$ Sister: 5 marbles; brother: 10 marbles; left marbles: 15.

4. A farmer bought a horse, cow, and pig for \$90. If he paid three times as much for the cow as for the pig, and five times as much for the horse as for the pig, what was the price of each?

   Solution: The initial value x is the pig, the cow $= 3x$, the horse $= 5x$.
   $\overbrace{x}^{\text{pig}} + \overbrace{3x}^{\text{cow}} + \overbrace{5x}^{\text{horse}} = 90 \Leftrightarrow 9x = 90 \overset{:9}{\Leftrightarrow} x = 10$. $3x = 30$, $5x = 50$. Check: $90 = 10 + 30 + 50\checkmark$ Horse: \$50, cow: \$30, pig: \$10.

5. A had seven times as many apples, and B three times as many as C had. If they all together had 55 apples, how many had each?

   Solution: C is the initial value. "A had seven times as many apples [...] as C": $A = 7 \cdot C$. "[...] and B three times as many as C had.": $B = 3 \cdot C$. $55 = A + B + C = 7C + 3C + C = 11C$. $55 = 11C \overset{:11}{\Leftrightarrow} C = 5$. $A = 7 \cdot C = 7 \cdot 5 = 35$. $B = 3 \cdot C = 15$. Check: $55 = 35 + 15 + 5\checkmark$ A=35, B=15, C=5.

6. The difference between two numbers is 36, and one is four times the other. What are the numbers?

Solution: Here, difference means a subtraction. $36 = 4x - x \Leftrightarrow 36 = 3\mathrm{x} \overset{:3}{\Leftrightarrow} x = 12$. $4x = 48$. Check: $36 = 48 - 12 = 36$✓ 12; 48

7. In a company of 48 people there is one man to each five women. How many are there of each?

   Solution: Men and women always appear in a ratio of 1:5, resp. men and women always appear in a kind of a unit: When there is one men, then there are also 5 women, we can describe this unit as (1+5). The total number of company members consists of x times such units: $x \cdot (1+5)$ resp. $48 = x(1+5)$. $48 = x \cdot 1 + x \cdot 5 \Leftrightarrow 48 = 6x \overset{:6}{\Leftrightarrow} x = 8$. The number of men is $x \cdot 1 = 8$, the number of women is $x \cdot 5 = 8 \cdot 5 = 40$. Check: $48 = 8 + 40$✓ Men: 8, women: 40.

8. A man left $1400 to be distributed among three sons in such a way that James was to receive double what John received, and John double what Henry received. How much did each receive?

   Solution: Henry is the initial value, x. John gets double what Henry receives, $2 \cdot x$, and James receives double what John receives, $2 \cdot (2x)$. The equation is $x + 2x + 2(2x) = 1400$. $7x = 1400 \overset{:7}{\Leftrightarrow} x = 200$. $2x = 400$. $4x = 800$. Check: $200 + 400 + 800 = 1400$✓ James: $800, John: $400, Henry: $200.

9. A field containing 45,000 feet was divided into three lots so that the second lot was three times the first, and the third twice the second. How large was each lot?

   Solution: Here, the initial value is the first lot, x. The second lot is "three times the first", $3x$. The third is "twice the second", $2 \cdot (3x)$. Equation: $x + 3x + 2(3x) = 45000$ resp. $10x = 45000 \overset{:10}{\Leftrightarrow} x = 4500$. 1^{st}: 4500. 2^{nd}: $3x = 3 \cdot 13500$. 3^{rd}: $6x = 6 \cdot 4500 = 27000$. Check: $4500 + 13500 + 27000 = 45000$✓.

   1^{st} lot: $4500, 2^{nd} lot: $13500, 3^{rd} lot: $27000.

10. There are 120 pigeons in three flocks. In the second there are three times as many as in the first, and in the third as many as in the first and second combined. How many pigeons in each flock?

    Solution: The first flock is the initial value resp. unknown variable, x. The second flock contains "three times as many as in the first", $3x$, the third "as many as in the first and second combined [addition!]", $x + 3x$. Equation: $x + 3x + (x + 3x) = 120$ resp. $8x = 120 \overset{:8}{\Leftrightarrow} x = 15$. First flock: 15. Second flock $3x$ resp. 45. Third flock: $4x$ resp. 60. Check: $15 + 45 + 60 = 120$✓. 1^{st} flock: 15, 2^{nd} flock: 45, 3^{rd} flock: 60.

11. Divide 209 into three parts so that the first part shall be five times the second, and the second three times the third.

    Solution: